Math 210C Lecture 4 Notes

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1 Noether's Normalization Lemma and The Nullstellensatz

1.1 Noether's normalization lemma

Let K be an algebraically closed field, $n \geq 1$, and $R = K[X_1, \dots, x_n]$. We had a map $V : \mathscr{P}(R) \to \mathscr{P}(K^n)$ defined by $V(S) = \{a = (a_1, \dots, a_n) \in K^n : f(a) = 0 \, \forall f \in S\}$, the vanishing locus of S. We also had a map $I : \mathscr{P}(K^n) \to \mathscr{P}(R)$ defined by $I(Z) = \{f \in R : f(a) = 0 \, \forall a \in Z\}$. The image of I is the radical ideals of R.

Recall that a subset Z of K^n is algebraic if $Z = B(S)$ fo rsome $S \subseteq R$, and we said last time that the algebraic sets form the closed sets in a topology on K^n the Zariski topology. we write $\mathbb{A}^n_K$ for K^n with this topology. $\mathfrak{m} = (x_1 - a_1, \dots, x_n - a_n)$, where $a_1, \dots, a_n \in K$, is a maximal ideal with vanishing locus $V(\mathfrak{m}) = \{(a_1, \dots, a_n)\}$. So points are closed. We also showed that $I(\{a_1, \dots, a_m\}) = (x_1 - a_1, \dots, x_n - a_n)$.

Lemma 1.1 (Noether's normalization lemma)**.** *Let F be a field. Let A be a finitely generated commutative F-algebra with generators $z_1, \dots, z_r \in A$. Then there exist $s \leq r$ and F-algebraically independent elements $t_1, \dots, t_s \in A$ such that A is integral over $F[t_1, \dots, t_s]$.*

Proof. Proceed by induction on r. If $r = 0$, we are done. Suppose $r \geq 1$. If $z_1, \dots, z_r$ are algebraically independent over F, we are done, so assume not. So there exists some nonzero $f \in F[x_1, \dots, x_r]$ such that $f(z_1, \dots, z_r) = 0$. Let d be the degree of f (i.e. the maximum of degrees of monomials, where the degree of a monomial is the sum of the degrees in each x_i). Without loss of generality, we may assume f is nonconstant as a polynomial in x_r. Let $g(x_1, \dots, x_r) = f(x_1 + x_r^{d+1}, x_2 + x_r^{(d+1)^2}, \dots, x_{r-1} + x_r^{(d+1)^{r-1}}, x_r) \in F[x_1, \dots, x_1]$. f is a sum of monomials, and for each monomial we get a sum of monomials contributing to g. Exactly one of these has the form (nonzero const. in F) $\cdot \, x_r^{\text{some power}}$. Since f has degree d and $(d+1)^i > d(d+1)^{i-1}$, each one of these terms for the various monomials in f has different degree. Every other monomial term that occurs in g has lower degree in x_r than the degree of the highest degree monomial cx_r^N with $c \in F^\times$. In other words, $g = cx_r^N +$ lower degree terms in x_r.

Set $w_i = z_i - z_r^{(d+1)^i}$ for $1 \leq i \leq r-1$. Then $g(w_1, \dots, w_r, z_r) = f(z_1, \dots, z_r) = 0$. So z_r is integral over $B = F[w_1, \dots, w_{r-1}]$. By induction, there exists $s \leq r-1$ and elements

$t_1, \dots, t_s \in B$ such that $t_1, \dots, t_s$ are F-algebraically independent and B is integral over $F[t_1, \dots, t_s]$. $A = B[z_r]$ is integral over B, so A is integral over $F[t_1, \dots, t_s]$. □

1.2 The weak Nullstellensatz

Theorem 1.1 (weak Nullstellensatz)**.** *Let K be algebraically closed. Every maximal ideal of $R = K[x_1, \dots, x_n]$ has the form $(x_1 - a_1, \dots, x_n - a_n)$ with $a_1, \dots, a_n \in K$.*

Proof. Let $\mathfrak{m} \subseteq R$ be a maximal ideal. Then $L = R/\mathfrak{m}$ is a field. L is a field extension of K and is finitely generated as a K-algebra (generated by images of $x_1, \dots, x_n$). By Noether's normalization lemma, there exist algebraically independent $t_1, \dots, t_s \in L$ (over K) such that L is integral over $K[t_1, \dots, t_s]$. If $s \geq 1$, then t_1^{-1} is integral over $K[t_1, \dots, t_s]$, which is not so. So $s = 0$, and L/K is therefore algebraic. Since K is algebraically closed, $L = K$.

Let a_i be the image of x_i under the quotient map $R \to R/\mathfrak{m} = K$. Then $x_i - a_i \in \mathfrak{m}$ for all i. So $\mathfrak{m} \subseteq (x_1 - a_1, \dots, x_n - a_n)$, so they are equal, as the latter ideal is maximal. □

1.3 Hilbert's Nullstellensatz

Theorem 1.2 (Hilbert's Nullstellensatz)**.** *I and V provide mutually inverse, inclusion reversing bijections $\{$radical ideals of $K[x_1, \dots, x_n]\} \leftrightarrow \{$algebraic sets in $\mathbb{A}_K^n\}$.*

Proof. Check that I and V are inclusion reversing. If $Z \subseteq \mathbb{A}_K^n$, then $V(I(Z)) \supseteq \overline{Z} \supseteq Z$. If$Z = V(\mathfrak{a})$ for some ideal $\mathfrak{a} \subseteq R$, then $V(I(Z)) = V(I(V(\mathfrak{a}))) \subseteq V(\mathfrak{a}) = Z$. Since $Z \subseteq V(I(Z))$, we get $V(I(Z)) = Z$.

If $\mathfrak{a}$ is an ideal of R, then $I(V(\mathfrak{a})) \supseteq \sqrt{\mathfrak{a}}$ from what we have already said. It remains to show that $I(V(\mathfrak{a})) \subseteq \sqrt{\mathfrak{a}}$ for all ideals $|mfa \subseteq R$. Let $f \in I(V(\mathfrak{a}))$. R is noetherian, so $\mathfrak{a} = (g_1, \dots, g_k)$, where $g_1, \dots, g_k \in R$. Let y be an indeterminate, so $R[y] = K[x_1, \dots, x_n, y]$. Let $J = \mathfrak{a}R[y] + (1 - fy) = (g_1, \dots, g_k, 1 - fy)$. If $z \in V(\mathfrak{a}) = V(g_1, \dots, g_k)$, then $(1 - fy)(z) = 1 - f(z)y = 1$ for all $z \in V(\mathfrak{a})$. Then $V(J) = \varnothing \subseteq \mathbb{A}_K^{n+1}$. We will finish this proof next time. □